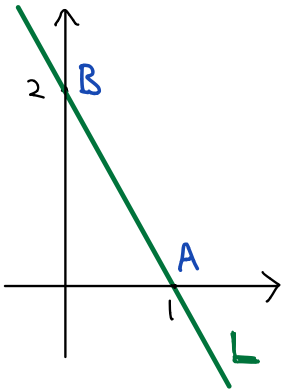
Math 2010 Week 2 (Au 1.3)

Linear Objects in $\mathbb{R}^n$

eg lines, plane, k-plane, hyperplane

Line

eg in $\mathbb{R}^2$



Equation form

$$2x+y=2$$

Parametric form

$$(x,y) = \overrightarrow{OA} + t\,\overrightarrow{AB}$$

$$= (1,0) + t(-1,2)$$

$$= (1-t,\ 2t)$$

Varying $t \in \mathbb{R}$ gives all the points on L

Note that

$$\begin{cases} x = 1-t \\ y = 2t \end{cases} \Rightarrow \begin{cases} t = \frac{x-1}{-1} \\ t = \frac{y}{2} \end{cases} \Rightarrow \frac{x-1}{-1} = \frac{y-0}{2}$$

Symmetric / Slope form

Parametric form of a line in $\mathbb{R}^n$

Let L be a line in $\mathbb{R}^n$

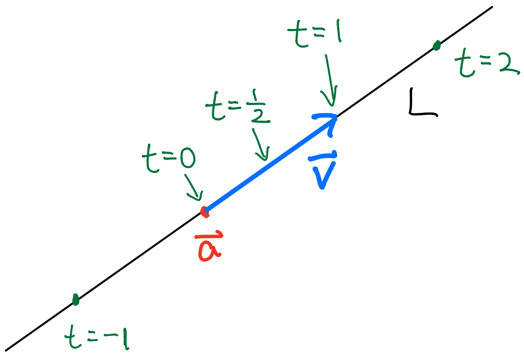
$\vec{a}$ is a point on L

$\vec{v}$ is a direction vector of L

Parametric form of L

$$\vec{x} = \vec{a} + t\vec{v}$$

$t \in \mathbb{R}$ is called a parameter



L is parametrized by $t \in \mathbb{R}$

eg A line L in $\mathbb{R}^3$ passes through

$$A=(1,2,3),\ B=(-1,3,5).$$

Then we can take $\vec{a}=(1,2,3)$,

$$\vec{v}=\overrightarrow{AB}=(-1-1,3-2,5-3)=(-2,1,2)$$

$\therefore$ Parametric form:

$$(x,y,z)=(1,2,3)+t(-2,1,2) \quad (*)$$

Rmk ① Parametric form is not unique.

Other parametrizations of L

$$(x,y,z)=(-1,3,5)+t(2,-1,-2)$$

$$\text{or}\quad (-1,3,5)+t(-4,2,4)$$

② From $(*)$, we get symmetric form

$$\frac{x-1}{-2}=\frac{y-2}{1}=\frac{z-3}{2}\quad (=t)$$

$$\Leftrightarrow \begin{cases} x-1=-2(y-2) \\ x-1=-(z-3) \end{cases}$$

Planes in $\mathbb{R}^3$

A plane P in $\mathbb{R}^3$ can be determined by

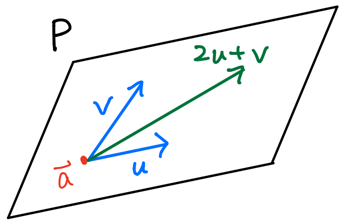
① 3 non-colinear points on P; or

② A point on P and 2 linearly independent directions (not same or opposite); or

③ A point on P and a normal vector

For ②

Let $\vec{a}\in P$, $\vec{u},\vec{v}$ are lin. indept directions of P



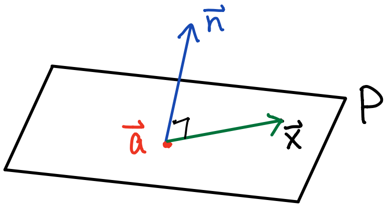
Parametric form of P

$$\vec{x}=\vec{a}+s\vec{u}+t\vec{v}$$

a point on P ($\vec{a}$); parameters (s, t); directions ($\vec{u}$, $\vec{v}$)

For ③, suppose

- $\vec{a} = (a_1, a_2, a_3)$ is a point on P
- $\vec{n} = (n_1, n_2, n_3)$ is a normal vector of P



Let $\vec{x} = (x, y, z) \in \mathbb{R}^3$. Then

$$\vec{x} \text{ is on } P \iff \vec{x} - \vec{a} \perp \vec{n}$$
$$\iff (\vec{x} - \vec{a}) \cdot \vec{n} = 0$$
$$\iff \vec{x} \cdot \vec{n} = \vec{a} \cdot \vec{n}$$

Equation of P

$$n_1 x + n_2 y + n_3 z = \overbrace{a_1 n_1 + a_2 n_2 + a_3 n_3}^{\text{constant}}$$

Rmk If $(a, b, c) \neq \vec{0}$, the equation

$$ax + by + cz = d$$

describes a plane in $\mathbb{R}^3$ with normal vector (a, b, c).

eg Suppose P is a plane passing through

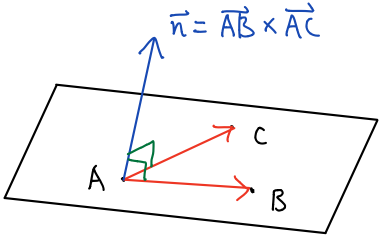
$$A = (0, 0, 1),\ B = (0, 2, 0)\ \ C = (-1, 1, 0)$$

Represent P using (i) parametric form ; (ii) equation

Sol (i)

$$\overrightarrow{AB} = (0, 2, 0) - (0, 0, 1)$$
$$= (0, 2, -1)$$

$$\overrightarrow{AC} = (-1, 1, 0) - (0, 0, 1)$$
$$= (-1, 1, -1)$$



$\therefore$ Parametric form:

$$(x, y, z) = (0, 0, 1) + s(0, 2, -1) + t(-1, 1, -1)$$

(ii) Take $\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 2 & -1 \\ -1 & 1 & -1 \end{vmatrix} = (-1, 1, 2) \perp P$

$\therefore$ Equation for P:

$$[(x, y, z) - (0, 0, 1)] \cdot (-1, 1, 2) = 0$$
$$(-1)x + (1)y + 2(z - 1) = 0$$
$$-x + y + 2z = 2$$

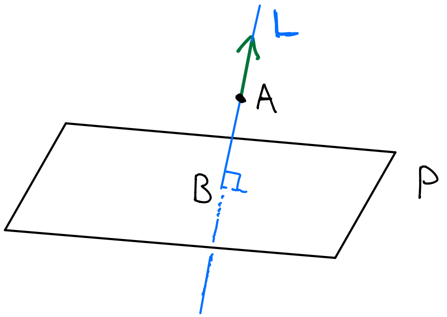
eg Find the distance between $A=(2,1,1)$ and the plane $P: -x+2y-z=-4$ ⊛

Sol From ⊛, $\vec{n}=(-1,2,-1) \perp P$

Consider the line L

$$\vec{x}(t)=\vec{A}+t\vec{n}=(2,1,1)+t(-1,2,-1)$$

Let $B=L\cap P$ be the intersection of L and P



Then B is the point on P closest to A

To find B, put

$\vec{x}(t)=(2-t,1+2t,1-t)$ into ⊛

Then $-(2-t)+2(1+2t)-(1-t)=-4$

$\Rightarrow 6t-1=-4 \Rightarrow 6t=-3 \Rightarrow t=-\frac{1}{2}$

$\therefore B=\vec{x}(-\frac{1}{2})=(\frac{5}{2},0,\frac{3}{2})$

Distance between A and P

$$=\|\overrightarrow{AB}\|=\sqrt{\left(\tfrac{5}{2}-2\right)^2+(0-1)^2+\left(\tfrac{3}{2}-1\right)^2}$$

$$=\frac{\sqrt{6}}{2}$$

Ex Find the distance between the lines

$$L_1(s)=(-4,9,-4)+s(4,-3,0)$$

$$L_2(t)=(5,2,10)+t(4,3,2)$$

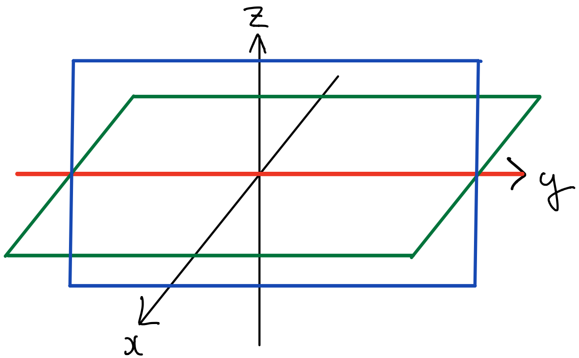
Hint: Find A on L_1, B on L_2 such that

$\overrightarrow{AB}\perp L_1, L_2$

(Ans: 13)

Line in $\mathbb{R}^3$ by equations

eg 1 y-axis in $\mathbb{R}^3$



One equation can only represent a plane.

We need at least two to represent a line.

eg. $\begin{cases} x = 0 \\ z = 0 \end{cases} \longleftrightarrow$ y-axis

System of equations $\longleftrightarrow$ Intersection of planes

Two ways to represent linear objects

$$\text{Equation(s)} \underset{\text{Eliminate parameters}}{\overset{\text{Gaussian Elimination}}{\rightleftarrows}} \text{Parametric form}$$

eg 2

Given equations of a line L

$$\begin{cases} x + y + 6z = 6 \\ x - y - 2z = -2 \end{cases} \longrightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ -4 \\ 1 \end{bmatrix} \quad (*)$$

Conversely, from (*)

$$\left.\begin{array}{l} x = 2 - 2t \\ y = 4 - 4t \end{array}\right\} \Rightarrow 2x - y = 0$$

$$\left.\begin{array}{l} y = 4 - 4t \\ z = t \end{array}\right\} \Rightarrow y = 4 - 4z \Rightarrow y + 4z = 4$$

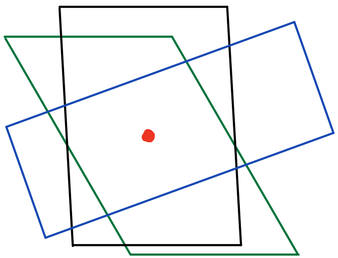
$\therefore$ Another set of equations for L: $\begin{cases} 2x - y = 0 \\ y + 4z = 4 \end{cases}$

eg 3 Three non-trivial equations of the form $ax+by+cz=d$ (in $\mathbb{R}^3$)

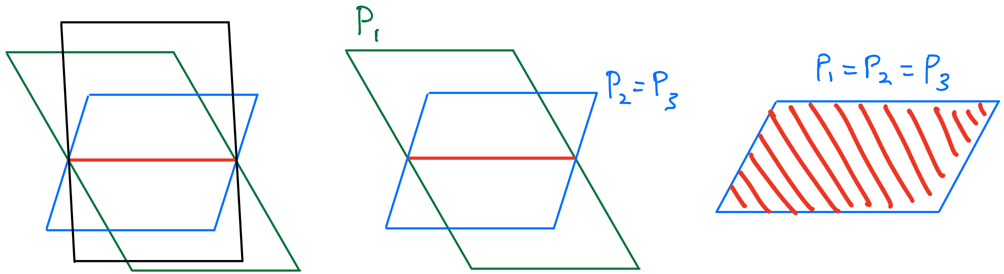
means $(a,b,c)\neq(0,0,0)$

Pictures

Case 1: Unique solution

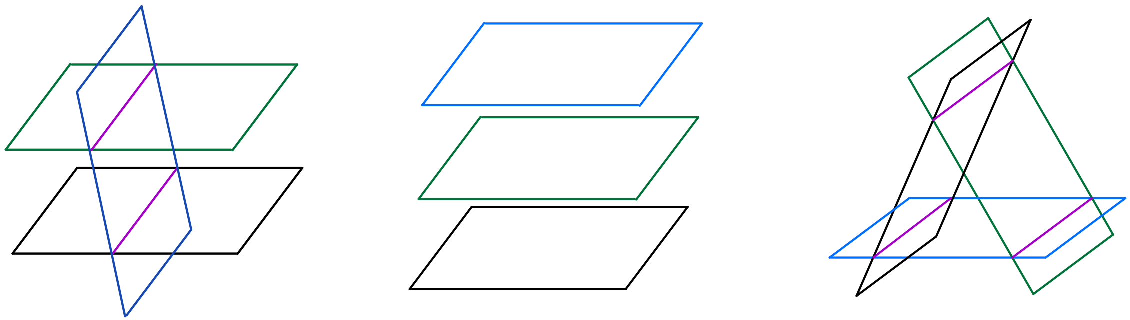


Case 2: Infinitely many solutions



Case 3: No solution $\leftrightarrow$ No common intersection

eg



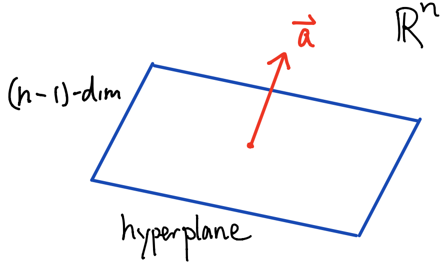
General linear objects in $\mathbb{R}^n$

- Similar to lines in $\mathbb{R}^3$, we need a system of equations to describe a plane in $\mathbb{R}^n$

- In $\mathbb{R}^n$, an equation of the form

$$\vec{a}\cdot\vec{x} = a_1x_1 + a_2x_2 + \cdots + a_nx_n = c, \quad \vec{a} \neq \vec{0}$$

describes a hyperplane (dimension $= n-1$) with normal vector $\vec{a}$



- To describe a k-dimensional plane P (called k-plane) in $\mathbb{R}^n$:

Parametric form

$$\vec{x} = \vec{q} + \sum_{i=1}^{k} t_i\vec{v}_i, \text{ where}$$

- $\vec{q} \in P$
- $\vec{v}_1, \cdots \vec{v}_k$ are k lin. indept vectors $/\!/ \; P$
- $t_1, \cdots, t_k$ are parameters

Equations

rank $A = n-k$ $\Updownarrow$

$n-k$ non-redundant equations $(i = 1, 2, \cdots, n-k)$

$$\sum_{j=1}^{n} a_{ij}x_j = c_i$$

System of $n-k$ equations $\longleftrightarrow$ Intersection of $n-k$ hyperplanes

Curves in $\mathbb{R}^n$ (Au 2.1, 2.2, Thomas 11.1, 11.2)

Defn Let $I \subset \mathbb{R}$ be an interval

A <u>curve</u> in $\mathbb{R}^n$ is a continuous function

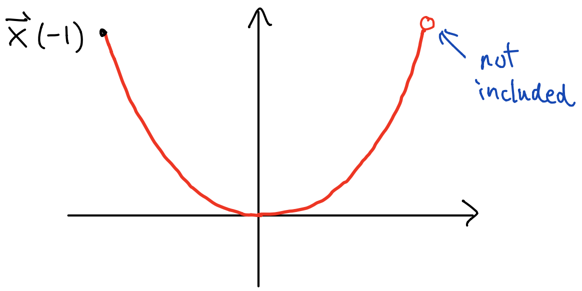
$$\vec{x} : I \longrightarrow \mathbb{R}^n$$

i.e $\vec{x}(t) = (x_1(t), x_2(t), \cdots, x_n(t))$ such that every component function $x_i(t)$ is continuous.

eg 1. $\vec{x} : [-1, 1) \longrightarrow \mathbb{R}^2$

$$\vec{x}(t) = (\underbrace{t}_{x}, \underbrace{t^2}_{y}) \Rightarrow x^2 = t^2 = y$$

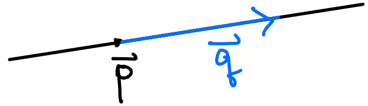
$\therefore$ Parabola



eg2 $\vec{x} : \mathbb{R} \to \mathbb{R}^3$, $\vec{p}, \vec{q} \in \mathbb{R}^3$, $\vec{q} \neq \vec{0}$

$$\vec{x}(t) = \vec{p} + t\vec{q}$$

(a straight line)

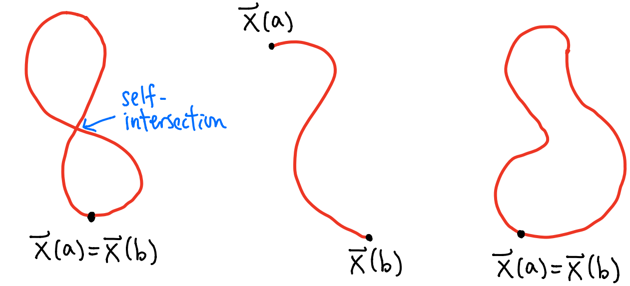


Defn A curve $\vec{x} : [a, b] \to \mathbb{R}^n$ is said to be

i. <u>closed</u> if $\vec{x}(a) = \vec{x}(b)$

ii. <u>simple</u> if $\vec{x}(t_1) \neq \vec{x}(t_2)$ for any $a \le t_1 < t_2 \le b$, except possibly at $t_1 = a$, $t_2 = b$

eg



Closed, not simple | Simple, not closed | Simple, closed

Thm Let $\vec{x}(t) = (x_1(t), x_2(t), \cdots, x_n(t))$. Then

① $\lim_{t\to a} \vec{x}(t) = \left(\lim_{t\to a} x_1(t), \cdots, \lim_{t\to a} x_n(t)\right)$

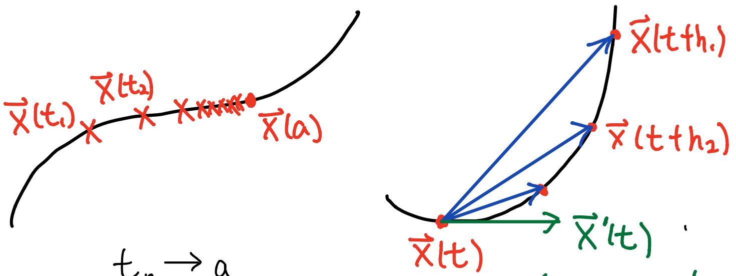
② $\vec{x}'(t) \overset{\text{Defn}}{=} \lim_{h\to 0} \dfrac{\vec{x}(t+h) - \vec{x}(t)}{h}$

$= (x_1'(t), x_2'(t), \cdots, x_n'(t))$

provided the limits exist.

Rmk $\vec{x}'(a)$ = tangent vector of $\vec{x}(t)$ at $t=a$.

Picture



$t_n \to a$

$x(t_n) \to x(a)$

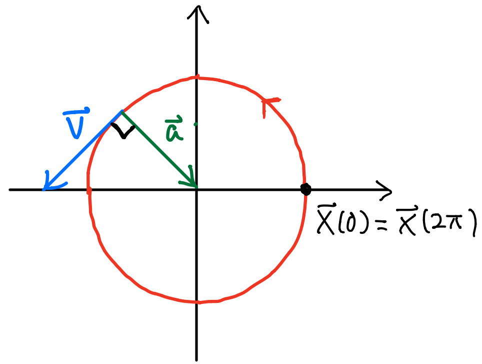
Physics

If $\vec{x}(t)$ = displacement at time t

Then $\vec{x}'(t)$ = velocity $= \vec{v}(t)$

$\vec{x}''(t)$ = acceleration $= \vec{a}(t)$

eg $\vec{x}(t) = (\cos t, \sin t) \quad 0 \le t \le 2\pi$



$\vec{v}(t) = \vec{x}'(t) = (-\sin t, \cos t) \perp \vec{x}(t)$

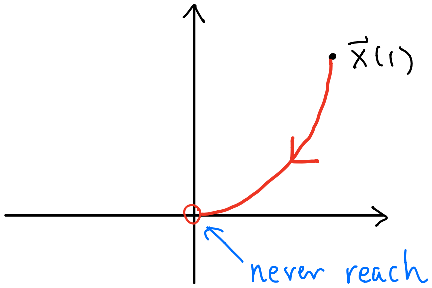
$\vec{a}(t) = \vec{x}''(t) = (-\cos t, -\sin t) = -\vec{x}(t)$

Also speed $= \|\vec{v}(t)\| = 1$

eg $\vec{x} : [1,\infty) \longrightarrow \mathbb{R}^2$

$$\vec{x}(t) = \left(\tfrac{1}{t}, \tfrac{1}{t^2} \right)$$

$$\underset{x}{\shortparallel} \quad \underset{y}{\shortparallel} \Rightarrow y = x^2$$



$$\lim_{t\to\infty} \vec{x}(t) = \left(\lim_{t\to\infty} \frac{1}{t}, \lim_{t\to\infty} \frac{1}{t^2} \right)$$

$$= (0, 0)$$

Rules

Let $\vec{x}(t)$, $\vec{y}(t)$ be curves in $\mathbb{R}^n$, $c \in \mathbb{R}$ be a constant

$f(t)$ be a real-valued function.

① $(\vec{x}(t) \pm \vec{y}(t))' = \vec{x}'(t) \pm \vec{y}'(t)$

② $(c\,\vec{x}(t))' = c\,\vec{x}'(t)$

③ $(f(t)\vec{x}(t))' = f'(t)\vec{x}(t) + f(t)\vec{x}'(t)$

④ $(\vec{x}(t) \cdot \vec{y}(t))' = \vec{x}'(t) \cdot \vec{y}(t) + \vec{x}(t) \cdot \vec{y}'(t)$

⑤ If $n=3$,

$$(\vec{x}(t) \times \vec{y}(t))' = \vec{x}'(t) \times \vec{y}(t) + \vec{x}(t) \times \vec{y}'(t)$$

Rmk ③ – ⑤ are product rules

Arclength

Let $\vec{x}(t)$ be a curve with

$\vec{x}'(t)$ exists and continuous

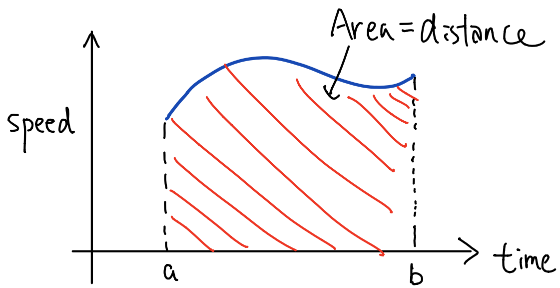
$$\boxed{\text{Arclength of } \vec{x}(t) \text{ for } a \le t \le b \text{ is} \quad s = \int_a^b \|\vec{x}'(t)\|\, dt}$$

If $\vec{x}(t)$ = displacement at time t

then $\vec{x}'(t)$ = velocity

$\|\vec{x}'(t)\|$ = speed

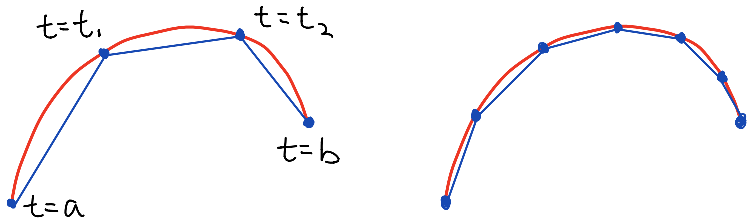
$\int_a^b \|\vec{x}'(t)\|\, dt$ = distance travelled



From Point of view of Mathematics

Approximate a curve by line segments :

Take $a = t_0 < t_1 < t_2 < \cdots < t_n = b$



Approximation　　　　Better approximation

$$s \approx \sum_{i=1}^{n} \|\vec{x}(t_i) - \vec{x}(t_{i-1})\|$$

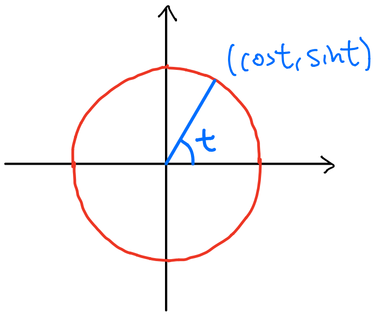
$$\left(\text{Recall: } \quad x'(t_i) = \lim_{t \to t_i} \frac{x(t) - x(t_i)}{t - t_i} \right)$$

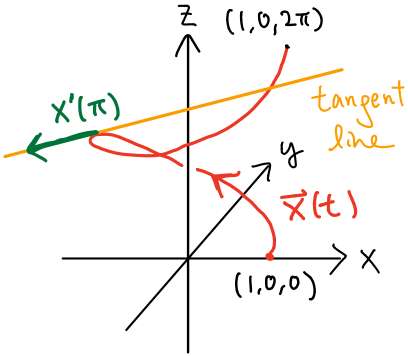
$$\approx \sum_{i=1}^{n} \|\vec{x}'(t_i)\| (t_i - t_{i-1})$$

Take Limit $\Rightarrow$ $s = \int_a^b \|\vec{x}'(t)\|\, dt$

eg (Helix)

$\vec{x}(t) = (\cos t, \sin t, t), \ t \in [0, 2\pi]$





a. Find the tangent line of $\vec{x}$ at $t = \pi$.

b. Find arclength of the helix.

Sol

a. $\vec{x}(t) = (\cos t, \sin t, t)$

$\vec{x}\,'(t) = (-\sin t, \cos t, 1)$

$\vec{x}\,'(\pi) = (0, -1, 1)$ ← direction of tangent

Also, $\vec{x}(\pi) = (-1, 0, \pi)$ ← a point on tangent line

∴ Parametric form of tangent line

$$\vec{x} = (-1, 0, \pi) + t(0, -1, 1)$$

b. $\|\vec{x}\,'(t)\| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2}$

$= \sqrt{2}$

$\Rightarrow \ s = \int_0^{2\pi} \|\vec{x}\,'(t)\| \, dt$

$= \int_0^{2\pi} \sqrt{2} \, dt$

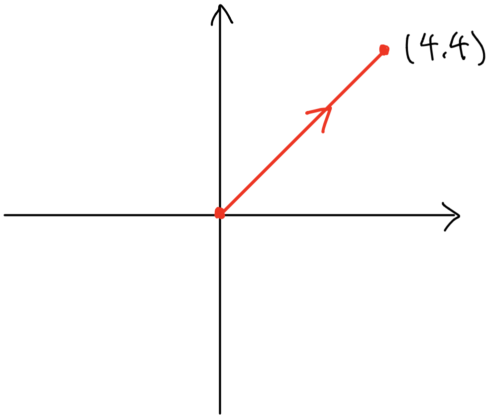
$= \left[\sqrt{2}\, t\right]_0^{2\pi}$

$= 2\sqrt{2}\,\pi$

Arclength is independent of parametrization

eg $\vec{x}(t) = (t, t)$ $\quad 0 \le t \le 4$

$\vec{y}(t) = (t^2, t^2)$ $\quad 0 \le t \le 2$



$\vec{x}, \vec{y}$ are two parametrizations of the same line segment

$\vec{x}'(t) = (1,1)$ $\qquad \vec{y}'(t) = (2t, 2t)$

arclength of $\vec{x}(t)$

$$= \int_0^4 \|\vec{x}'(t)\|\, dt$$

$$= \int_0^4 \sqrt{2}\, dt$$

$$= 4\sqrt{2}$$

arclength of $\vec{y}(t)$

$$= \int_0^2 \|\vec{y}'(t)\|\, dt$$

$$= \int_0^2 \sqrt{(2t)^2 + (2t)^2}\, dt$$

$$= \int_0^2 2\sqrt{2}\, t\, dt$$

$$= \left[\sqrt{2}\, t^2\right]_0^2$$

$$= 4\sqrt{2}$$

Same answer

Rmk

To prove arclength is independent of parametrization use change of variable for integration (Ex)